Diffusive Limits on the Penrose Tiling

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Abstract In this paper random walks on the Penrose lattice are investigated. Heat kernel estimates and the invariance principle are shown.

1 Introduction

The Penrose tiling [9, 15] is the most famous nonperiodic tiling of the plane and is an unfailing source of beautiful properties and phenomena to be explored [1]. There is associated with the Penrose tiling by the usual duality relation a lattice that may be called the Penrose lattice. Kunz [12] discussed conditions under which simple nearest-neighbor random walk on the Penrose lattice would conform to a version of the central limit theorem, but did not completely resolve the issue. In the present paper, it is proved that random walk on the Penrose lattice satisfies the invariance principle (in the annealed sense, see detailed explanation in the next section) that with appropriate scalings, the random walk process converges weakly to a non-degenerate rotation-invariant Brownian motion. Szász [17] has conjectured that a related Lorentz scatter system also satisfies this principle. The proof of the discrete, random walk counterpart of the conjecture, also formulated by Szász, is the main result of the paper.

Central to the approach followed here is the notion of roughly isometric weighted graphs, defined precisely later, which share diffusion properties (cf. [10]). This enables us to relate the simple random walk on the Penrose lattice to translation invariant walks on the plane square lattice $\mathbb{Z}^2$. Let $d(x, y)$ denote the graphical distance (number of edges in the shortest path) between vertices x and y. Many random walk processes obey a two-sided Gaussian

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heat kernel estimate $(GE_{\alpha,2})$

$$\frac{c}{n^{\alpha/2}} \exp\left(-\frac{Cd(x,y)^2}{n}\right) \leq \widetilde{p}_n(x,y) \leq \frac{C}{n^{\alpha/2}} \exp\left(-\frac{cd(x,y)^2}{n}\right) \tag{1}$$

for $0 < n < d(x,y)$, with $\alpha \geq 1$, $c > 0$ and $C > 0$. Here $\widetilde{p}_n = p_n + p_{n+1}$, where $p_n(x,y)$ is the probability that a walker departing from vertex x is found at vertex y after n steps. Delmotte [6] has shown that the bounds (GE_2):

$$\frac{c}{V(x,\sqrt{n})} \exp\left(-\frac{Cd(x,y)^2}{n}\right) \leq \widetilde{p}_n(x,y) \leq \frac{C}{V(x,\sqrt{n})} \exp\left(-\frac{cd(x,y)^2}{n}\right), \tag{2}$$

(where $V(x,r)$ denotes the volume of geodesic balls in graph distance) are stable under rough isometry for random walks on weighted graphs (which we define precisely later). In other words if two graphs are roughly isometric and (2) holds for one then it holds for the other as well (see also [18]). We know that (2) holds for the simple symmetric random walk on $\mathbb{Z}^2$ ($V(x,r) \simeq r^2$, i.e. satisfy (1) with $\alpha = 2$) and consequently holds for the Penrose graph if it that graph is roughly isometric to $\mathbb{Z}^2$. A very short, direct proof will be given of rough isometry between the Penrose graph and $\mathbb{Z}^2$.

Unfortunately the exponents in the upper and lower estimates in (1) and (2) contain different constants which reflects local inhomogeneities. The rough isometry invariance and the Gaussian estimate (2) have been proved along a series of estimates in which some cumulation of constants is unavoidable. Consequently if we are looking for the central limit theorem or for the invariance principle we need a different approach.

In the field of stochastic processes and statistical physics a powerful method has been developed to investigate random walks in random environment, on percolation clusters and interacting particle systems. A key result in this direction is the celebrated paper by Kipnis and Varadhan [11] and its influential extension by De Masi, Ferrari, Goldstein, Wick [5] in which annealed central limit theorem and invariance principle is shown. (The initial environment is averaged with respect to an invariant measure, for further details see Sect. 4 and [16] on convergence notions in random environments). That result provides us immediate derivation of the central limit theorem and the invariance principle for the random walk on the Penrose lattice in the same sense.)

In what follows we introduce the basic terminology then the statement is proved.

2 Preliminaries

We will consider infinite connected graphs with vertex set Γ. If an edge joins vertices x and y we denote that edge by $x \sim y$. The distance $d(x,y)$ will be the shortest path metric. In particular we will speak about the integer lattice $\mathbb{Z}^m = (\mathbb{Z}^m, d)$ graph where vertexes are elements of $\mathbb{Z}^m$ and $x, y \in \mathbb{Z}^m$ form an edge, $x \sim y$, if and only if $|x - y|$. We will speak about the integer lattice $(\mathbb{Z}^m, |\cdot|)$ if we consider the same vertex and edge set but the metric is the Euclidean one. We do not define the Penrose tiling (cf. [4]). We assume that it is well defined and given for us on $\mathbb{R}^2$.

The Penrose lattice $(\Gamma, |\cdot|)$ is a metric space. It is the set of *centers* (centroids) of the tiles equipped with the Euclidean distance. Two tiles are neighbors if they are edge adjacent. Two vertexes of the Penrose lattice are neighbors if they centers of neighboring tiles. Those vertexes form edges of the lattice. We will speak about Penrose graph, with the same vertex

and edge set but with $d(x, y)$, the shortest path graphs distance. Let $\Gamma = (\Gamma, d_P)$ denote the Penrose graph, and $(\mathbb{Z}^2, d_{\mathbb{Z}})$ integer lattice graph.

We distinguish tilings by fixing a reference vertex and identifying it with the origin of $\mathbb{R}^2$. We denote by Ω the space of tilings (union of ten tori $\Omega_{i,.j} \subset \mathbb{R}^2$ $i, j = 0..4, i < j$, for details cf. Sect. 2.1 in [12]). Let $d(x)$ be the degree of x, the number of neighbors.

Definition 1 A graph is weighted if a symmetric weight function $\mu_{x,y} > 0$ is given on the edges. This weight defines a measure on vertexes and sets:

$$\mu(x) = \sum_{y \sim x} \mu_{x,y}$$

$$\mu(A) = \sum_{x \in A} \mu(x).$$

We denote the ball of radius of r by $B(x, r) = \{y : d(x, y) < r\}$ and we call $V(x, r) = \mu(B(x, r))$ its volume. In particular for the Penrose lattice (and graph) $\mu_{x,y} \equiv 1$ if $x \sim y$ is an edge, while $\mu_{x,y}$ is zero otherwise. The same applies for the integer lattice.

A Markov chain, with transition probabilities $P(x, y)$ is reversible (μ-reversible) if there is a μ measure such that $\mu(x)P(x, y) = \mu(y)P(y, x)$.

Definition 2 In general a random walk X_n on Γ, a weighted graph, with μ is a reversible Markov chain defined by the one step transition probabilities:

$$P(X_n = y | X_{n-1} = x) = P(x, y) = \frac{\mu_{x,y}}{\mu(x)}.$$

The random walk on the Penrose lattice (and graph) is reversible Markov chain (c.f. [7]) with transition probability $P(x, y) = 1/d(x) = 1/4$ for $x \sim y$. It is clear that $d(x)P(x, y) = d(y)P(y, x) = 1$. Denote X_i the actual position of the Markov chain (random walk). The $X_0 = x$ site is the reference vertex of the Penrose tiling, associated to the origin.

3 Heat Kernel Estimate for the Penrose Graph

First of all we give the definition the bi-Lipschitz property and rough isometry.

Definition 3 A metric space (Γ, d) is bi-Lipschitz to (Γ', d') if there is a bijection Φ from Γ to Γ' and a constant $C > 1$ such that for all $x \neq y \in \Gamma$

$$\frac{1}{C}d(x, y) \leq d'(\Phi(x), \Phi(y)) \leq Cd(x, y). \tag{3}$$

Definition 4 Two weighted graphs Γ with μ and Γ' with μ' are roughly isometric (or quasi isometric) (cf. [3, Definition 5.9] or [10]) if there is a map ϕ from Γ to Γ' such that there are $a, b, c, M > 0$ for which

$$\frac{1}{a}d(x, y) - b \leq d'(\phi(x), \phi(y)) \leq ad(x, y) + b \tag{4}$$

for all $x, y \in \Gamma$,

$$d'(\phi(\Gamma), y') \leq M \tag{5}$$

for all $y' \in \Gamma'$ and

$$\frac{1}{c}\mu(x) \leq \mu'(\phi(x)) \leq c\mu(x) \tag{6}$$

for all $x \in \Gamma$.

Remark 1 It is clear that if ϕ from Γ to Γ' is a rough isometry then there is a rough isometry ϕ' from Γ' to Γ as well.

Theorem 1 (Solomon [14]) *The Penrose lattice is bi-Lipschitz to the integer lattice.*

Proposition 1 *The Penrose lattice is rough isometric to the integer lattice.*

The statement follows from the bi-Lipschitz property. A very short and direct proof can be given, which we present here.

Proof Denote by m the smaller distance between the opposite boundaries of the thin rhombus and write $\varepsilon = \sqrt{2}m/4$. Consider the integer lattice $\varepsilon\mathbb{Z}^2$. It is clear that if an open square with edge length ε contains a center of a rhombus that it is fully contained by the closed rhombus. Let Ψ map the center of the rhombus to the center of the square. It is clear that Ψ is rough isometry from the Penrose lattice to the integer lattice.   □

Proposition 2 *The Penrose graph is roughly isometric to the integer lattice graph.*

Proof Let us consider Ψ, the map introduced above, between Γ and $\varepsilon\mathbb{Z}^2$. Now we consider the graph distances $d_P, d_{\mathbb{Z}}$. It is clear that

$$d_P(x, y) \leq d_{\mathbb{Z}}(\Psi(x), \Psi(y)).$$

The opposite inequality is also easy. Let $2L$ be the maximal number of squares which is needed to cover the largest diagonal of rhombi. It is clear that L is bounded since the diameter of the rhombi is also bounded. Then

$$d_{\mathbb{Z}}(\Psi(x), \Psi(y)) \leq 2Ld_P(x, y).$$

It is also clear that the conditions (5), (6) are satisfied.   □

Lemma 1 *The Penrose lattice and the Penrose graph are roughly isometric.*

Proof Let Φ_1 the rough isometry from the graph Γ to the lattice Z^2, let Φ_2 the identity map on Z^2 which is bi-Lipschitz between the integer lattice and graph, finally Φ_3 the rough isometry from the integer graph to the Penrose graph. The existence of Φ_3 follows from Proposition 2 and Remark 1. Then $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$ is rough isometry between the Penrose lattice and graph.   □

Now we recall Delmotte's result [6] omitting the third equivalent statement, the parabolic Harnack inequality, since we do not need it in the sequel.

Theorem 2 *Let Γ with μ be a weighted graph. Assume that there is a $p_0 > 0$ such that for all edges $P(x, y) \geq p_0$. Then the following statements are equivalent.*

1. *There are $C, c > 0$, $\alpha \geq 1$ such that for all $x, y \in \Gamma$ and $n > 0$*

$$\frac{c}{V(x, \sqrt{n})} \exp\left(-C\frac{d(x, y)^2}{n}\right) \leq \widetilde{p}_n(x, y) \leq \frac{C}{V(x, \sqrt{n})} \exp\left(-c\frac{d(x, y)^2}{n}\right) \quad (7)$$

   holds,
2. (i) *The volume doubling condition (VD) holds: there is a $C > 0$ such that for all $x \in \Gamma$, $r \geq 1$*

$$V(x, 2r) \leq CV(x, r)$$

   and
   (ii) *the Poincare inequality (PI_2) holds: there is a $C > 0$, such that for all x and, $r > 1$, $f : B(x, r) \to \mathbb{R}$*

$$\sum_{y \in B(x,r)} (f(y) - f_B)^2 \mu(y) \leq cr^2 \sum_{y,z \in B(x,r)} (f(y) - f(z))^2 \mu_{y,z}$$

   where $f_B = \frac{1}{V(x,r)} \sum_{y \in B} f(y)\mu(y)$, $f \neq 0$, $B = B(x, r)$.

It is well known that the volume doubling property as well as the Poincare inequality are rough isometry invariant. Of course volume doubling can be replaced with $V(x, r) \simeq r^\alpha$ and (7) reduces into (1).

Corollary 1 *If Γ with μ and Γ' with μ' are roughly isometric graphs then $(GE_{\alpha,2})$ holds for one if and only if holds for the other.*

Theorem 3 *The Gaussian estimate $(GE_{2,2})$ holds for the random walk on the Penrose graph.*

Proof Proposition 1 ensures that Penrose graph is rough isometric to $\mathbb{Z}^2$. It is well-known that $(GE_{2,2})$ holds for the random walk on the integer lattice (graph), and then by Corollary 1 $(GE_{2,2})$ holds for the random walk on the Penrose graph as well. □

4 The Invariance Principle

In this section we confine ourself to the Penrose lattice. The Gaussian estimate $(GE_{2,2})$ provides a nice description of the random walk on the Penrose graph but the different constants in the exponents mean that we have only estimate of the variance and it may change from place to place as well as in time. Particularly we do not know if the properly scaled mean square displacement $\frac{1}{n}E(d^2(X_0, X_n))$ has a limit. Of course we expect that due to the asymptotic spherical symmetry of the Penrose tiling the diffusion matrix is the identity matrix up to a fixed constant multiplier. In other words the scaled mean square displacement is direction independent. In order to obtain the invariance principle for the Penrose lattice we need a different method. This is the method of ergodic processes of the environment "seen from the tagged particle" (cf. [5, 11]). Thanks to the result of De Masi et al. [5] it is enough

to check that the conditions of Theorem 2.1 in [5] are satisfied and that the covariance matrix is positive definite.

There are several formulation of the invariance principle, (for the classical formulation see [2]). We say that X_t satisfies the central limit theorem (*CLT*) if there is a $\sigma \geq 0$ such that

$$\frac{X_t}{\sqrt{t}} \to N(0, \sigma)$$

in distribution. What is slightly stronger, the process can be re-scaled, that is for $\varepsilon \to 0$ for all t

$$X^{(\varepsilon)} = \varepsilon X_{t/\varepsilon^2} \to W_\sigma(t) \tag{8}$$

where the convergence is in the sense of finite dimensional distributions and $W_\sigma(t)$ is the Wiener process with mean 0 and variance σ^2. The invariance principle holds if the convergence holds for the path-space measures for the processes (cf. Goldstein [8]). This requirement is equivalent with (8) and "tightness" of the process (cf. [2]).

Let $\mathbb{P}_\mu$ the path-space measure of the processes $(X_t^{(\varepsilon)})_{t\geq 0}, \varepsilon > 0$. Following [5] we say that $X^{(\varepsilon)} = (X_t^{(\varepsilon)})_{t\geq 0}$ converges weakly in μ-measure to Y if for any continuous function F on the path-space $D([0, \infty), \Gamma)$:

$$\mathbb{E}_\mu(F(X^{(\varepsilon)})) \to \mathbb{E}_\mu(Y).$$

Definition 5 We say that the invariance principle holds if the weak convergence in μ-measure to the Wiener process holds in this sense.

We consider the environment process ω_n seen from the particle. It is more convenient to use (as it is done by Kunz in [12]) the Markov chain $z_n = (\omega_n, X_n)$. Let us note that $\mathbb{E}_\mu(F(X^{(\varepsilon)}))$ means that the underlying environment process is started from the invariant measure μ, $\mathbb{E}_\mu(F(X^{(\varepsilon)})) = \mathbb{E}_\mu(F(X^{(\varepsilon)})|\omega_0 = \omega)$ ω is chosen according to μ, in other words we average with respect to the (initial) environment and obtain an annealed type of result (cf. [16]).

Theorem 4 *The random walk on the Penrose lattice satisfies the invariance principle with non-degenerate covariance matrix.*

Proof Kunz has shown that z_n is ergodic (see also a more general result by Robinson [13]). The invariant measure is combination of the Lebesgue measure λ on the tori $\Omega_{i,j}$. For $0 \leq i < j \leq 4$

$$\mu|_{\Omega_{i,j}} = \tau^{1-\lfloor\frac{j-i}{2}\rfloor}\lambda$$

where τ is the "Golden mean" $(\sqrt{5}+1)/2$. It is clear that $X_n = \sum_{i=1}^n V(z_{i-1}, z_i)$ where $V(z_{i-1}, z_i) = X_i - X_{i-1}$ is an antisymmetric function (cf. [5] (2.3), (2.6), and the remark below it.) It follows from the definition that X_n and z_n as well are reversible. The random walk is well defined for all ω, for all Penrose lattices, with given reference vertex, hence we have the path metric $\mathbb{P}_\omega$. Similarly $\mathbb{P}_\mu$ is well defined if the initial lattice is chosen according to the invariant measure. Let $\mathbb{E}_\mu$ denote the corresponding expected value. The only properties are to check that the conditional drift

$$\varphi = \mathbb{E}_\mu(X_1 - X_0|X_0) \tag{9}$$

exists and that the covariance matrix

$$D = \mathbb{E}_\mu((X_1 - \varphi)(X_1 - \varphi)^*) \tag{10}$$

is non-degenerate. For any given $X_0 = x$ the conditional drift evidently exists thanks to the bounded distances of neighbors. The conditions of the main result of [5] are satisfied, hence the invariance principle holds for X_i in the sense of Definition 5.

Let us recall that the D always exists (see Remark 1 below (2.30) in [5]).

We show that the covariance matrix is positive definite. Let us consider the annulus $B(0, C_2\sqrt{n}) \backslash B(0, C_1\sqrt{n})$ intersected with the cone about a given direction $e \in \mathbb{R}^2$ with angle α fixed $\pi/2 > \alpha > 0$. Let H denote the intersection. The constants C_1, C_2 are arbitrary and fixed. Let us recall (Lemma 1) that the Penrose lattice and graph are roughly isometric

$$\frac{1}{n}\mathbb{E}(e^* X_n X_n^* E e | X_0 = x_0)$$

$$= \frac{1}{n}\mathbb{E}\big((e^* X_n)^2 | X_0 = x_0\big)$$

$$\geq \frac{1}{n} \sum_{x \in H} (ex)^2 P_n(x_0, x)$$

$$\geq \frac{|H|}{n} c'(\cos(\alpha)\sqrt{n})^2 \frac{c''}{n} \exp\left[-C \frac{(aC_2\sqrt{n} + b)^2}{n}\right] \geq c > 0$$

independently of x_0 and e, hence the covariance matrix is non-degenerate. (Here the effect of all previous constants are absorbed into the last constant c.) By this we have shown that the invariance principle holds for the random walk on almost all Penrose lattice and the limiting process is a non-degenerate Brownian motion. □

Remark 2 The results presented in this paper carry over easily to other quasicrystals which can be constructed by the projection methods similar to the one produces the Penrose tiling. This applies to generalized Penrose tilings (produced by p-grids), higher dimensional Penrose tilings and stochastic tilings.

Remark 3 It seems plausible that with some extra work one can show that the covariance matrix is the identity matrix multiplied with a positive constant. The exact value of the constant ought to be determined as well.

References

1. Arnold, V.I.: Remarks on quasicrystallic symmetries. Physica D **33**, 21–25 (1988)
2. Billingsley, P.: Convergence of Probability Measures. Wiley, New York (1968)
3. Burago, D., Kleiner, B.: Rectifying separated nets. Geom. Funct. Anal. **12**, 1 (2002)
4. de Bruijn, N.G.: Algebraic theory of Penrose's nonperiodic tilings of the plane. Indag. Math. **43**(1), 39–66 (1981)

5. De Masi, A., Ferrari, P.A., Goldstein, S., Wick, D.W.: An invariance principle for reversible Markov processes with application to random motion in random environments. J. Stat. Phys. **55**, 787–855 (1989)
6. Delmotte, T.: Parabolic Harnack inequality and estimates of Markov chains on graphs. Rev. Math. Iberoam. **15**, 181–232 (1999)
7. Doyle, P., Snell, L.: Random Walks and Electric Networks. MAA Carus Monographs (1984)
8. Goldstein, S.: Antisymmetric functionals of reversible Markov processes. Ann. Inst. Henri Poincaré Probab. Stat. **31**(1), 177–190 (1995)
9. Grünbaum, B., Shephard, G.C.: Tilings and Patterns. W.H. Freeman, New York (1987)
10. Hambly, B.M., Kumagai, T.: Heat kernel estimates for symmetric random walks on a class of fractal graphs and stability under rough isometries. In: Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot. Proc. Symp. Pure Math., vol. 72, pp. 233–259 (2004). Part 2
11. Kipnis, C., Varadhan, S.R.S.: Central limit theorem for additive functionals of reversible Markov processes and application to simple exclusions. Commun. Math. Phys. **104**, 1–19 (1986)
12. Kunz, M.: Random walks and localization on the Penrose lattice. Physica B **293**, 164–182 (2000)
13. Robinson, E.A. Jr.: The dynamics of Penrose tilings. Trans. Am. Math. Soc. **384**, 4447–4464 (1996)
14. Solomon, Y.: http://arxiv.org/abs/0711.3707v1
15. Senechal, M.: Quasicrystals and Geometry. Cambridge University Press, Cambridge (1996)
16. Sidoravicius, V., Sznitman, A.S.: Quenched invariance principles for walks on clusters of percolations or among random conductances. Probab. Theory Relat. Fields **129**(2), 219–244 (2004)
17. Szász, D.: Some challenges in the theory of (semi)-dispersing billiards. Nonlinearity **21**, T187–T193 (2008)
18. Telcs, A.: The Art of Random Walks. Lecture Notes in Mathematics, vol. 1885. Springer, Berlin (2006)